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MAGNETIC FIELDS AND PASSIVE SCALARS IN POLYAKOV'S CONFORMAL TURBULENCE

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Abstract

Polyakov has suggested that two dimensional turbulence might be described by a minimal model of conformal field theory. However, there are many minimal models satisfying the same physical inputs as Polyakov's solution $\mathcal{M}_{(2,21)}$. Dynamical magnetic fields and passive scalars pose different physical requirements. For large magnetic Reynolds number other minimal models arise. The simplest one, $\mathcal{M}_{(2,13)}$, makes reasonable predictions that may be tested in the astrophysical context. In particular, the equipartition theorem between magnetic and kinetic energies does not hold: the magnetic one dominates at larger distances.

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Turbulence is one of the longstanding problems of theoretical physics. It affects an enormous variety of phenomena, from galactic plasma to cigarette smoke [1]. As such, the problem is far from being solved. The randomness of the basic physical quantities clearly suggests that the theory of random fields should be used to study this problem. However, since one is dealing with a strongly coupled field theory problem, the lack of reliable computational techniques has slowed down the progress.

Recently, Polyakov [2] has made a bold attempt to understand fully developed, homogeneous, isotropic, stationary two dimensional turbulence by using conformal field theory. Fully developed turbulence simply means that the motion of the fluid is not affected by external constraints such as boundaries (at large distances), or viscosity (at short distances). It is in this case that universal features of turbulence have a better chance to be understood. For fully developed turbulence, the assumptions of spatial homogeneity (translation invariance) and isotropy (rotation invariance) are very reasonable. More subtle is the stationary assumption (time translation invariance) because it implies that turbulence must be sustained by external forces. The restriction to the two dimensional case is, of course, because one wants to use the full power of the conformal field theory [3].

Polyakov's basic idea is to describe the universal features of two dimensional turbulence by fitting it into a (non unitary) minimal model $\mathcal{M}_{(p,q)}$ of two dimensional conformal field theory, here $1 < p < q$, p and q integers, prime to each other [3]. Any such model contains $(p-1)(q-1)/2$ primary fields $\Phi_{(m,n)}$, $0 < m < p$, $0 < n < q$, $\Phi_{(m,n)} \equiv \Phi_{(p-m,q-n)}$ with conformal weight

$$\Delta_{(m,n)} = \frac{(pn-qm)^2 - (p-q)^2}{4pq}. \tag{1}$$

For completeness, let us recall that the fusion rules are

$$[\Phi_{(m_1,n_1)}] \times [\Phi_{(m_2,n_2)}] = \sum_{k=|m_1-m_2|+1}^{\substack{\min(m_1+m_2-1,\\ 2p-m_1-m_2-1)}}{}' \quad \sum_{l=|n_1-n_2|+1}^{\substack{\min(n_1+n_2-1,\\ 2q-n_1-n_2-1)}}{}' \; [\Phi_{(k,l)}], \tag{2}$$

where the symbol Σ' means that the sum is performed with index jumping of two by two, e.g. $k = 3, 5, 7\cdots$. The infrared contributions from the regular terms must be always kept in mind because otherwise exact conformal symmetry would imply the orthogonality theorem between primary fields.

In two dimensions, the velocity field v_α of an incompressible fluid ($\partial_\alpha v_\alpha = 0$), can be written in terms of a single (pseudo) scalar function ψ, called the stream function:

$$v_\alpha = \epsilon_{\alpha\beta}\partial_\beta\psi. \tag{3}$$

The vorticity ω associated to the velocity field is a scalar and can also be written purely in terms on the stream function:

$$\omega = \epsilon_{\alpha\beta}\partial_\alpha v_\beta = -\partial_\alpha\partial_\alpha\psi. \tag{4}$$

One then attempts to interpret ψ as one of the primary fields of some minimal model $\psi = \Phi_{(m,n)}$ and to use the dynamics of the theory (Navier-Stokes equations plus energy consideration) to fix p, q, m and n. Let us briefly recall how this is done in [2].

The Navier-Stokes equations for this two dimensional incompressible fluid can be written, (after taking the curl) as

$$\frac{d}{dt}\omega \equiv \dot{\omega} + \epsilon_{\alpha\beta}\partial_\beta\psi\partial_\alpha\omega = \nu\partial_\alpha\partial_\alpha\psi. \tag{5}$$

The notation is as follows: throughout this paper, the symbol $\frac{d}{dt}$ will denote the convective derivative of any quantity, i.e., the time derivative along the motion of a particle in the fluid. The ordinary partial time derivative will be denoted by a dot or by ∂_t. The relation between the two is

$$\frac{d}{dt} = \partial_t + v_\alpha\partial_\alpha \equiv \partial_t + \epsilon_{\alpha\beta}\partial_\beta\psi\partial_\alpha. \tag{6}$$

The constant ν represents the kinematic viscosity. In the following we will always work in the so called inertial range, where the effective Reynolds number is huge and the viscous term in (5) can be neglected;

$$\frac{d}{dt}\omega = 0. \tag{7}$$

One then talks of (7) as the inviscid Navier-Stokes equations.

In two dimensions, in addition to the kinetic energy per unit mass density $E = \frac{1}{2}\int v_\alpha v_\alpha d^2x$, there is another quadratic conserved quantity. This new quantity is defined as the integral of the square vorticity per unit mass density: $H = \frac{1}{2}\int \omega^2 d^2x$ and it is known as the "enstrophy". From the detailed analysis in [4] it turns out that, for scales smaller than the scale at which energy is fed into the system, the flux of enstrophy η is the only relevant dimensional parameter and one could estimate the scaling behavior of all observables in the Kolmogorov approximation. For example, the energy spectrum $E(k)$, defined as

$$\frac{1}{2} < v_\alpha v_\alpha >= \int d^1 k E(k), \quad k = \sqrt{k_\alpha k_\alpha}, \tag{8}$$

is given, by dimensional analysis, as $E(k) \approx \eta^{2/3} k^{-3}$. Of course, the aim is to find corrections to this scaling behavior. From the definition of $E(k)$ and being v_α a level one spin one field in $[\psi]$ one has

$$E(k) \approx k^{-1+2(\Delta_v + \bar{\Delta}_v)} = k^{1+4\Delta_\psi}, \tag{9}$$

and what is left is to find Δ_ψ.

In order to achieve this goal, one first translates the constant flux of enstrophy into the scaling behavior of the two point correlation function:

$$< \dot{\omega}(r)\omega(0) > \approx r^0. \tag{10}$$

Now, if the primary field ψ satisfies the fusion rules

$$[\psi] \times [\psi] = [\phi] + \text{less singular terms}, \tag{11}$$

and ϕ is neither the identity nor a level two degenerate field, from (7) one has $\dot{\omega} \in [\phi]$ at level two and from (4) one has $\omega \in [\psi]$ at level one. Hence:

$$\Delta_\omega = \bar{\Delta}_\omega = \Delta_\psi + 1 \quad \text{and} \quad \Delta_{\dot{\omega}} = \bar{\Delta}_{\dot{\omega}} = \Delta_\phi + 2. \tag{12}$$

Equation (10) then can be used to fix the sum of the conformal weights:

$$\Delta_{\dot{\omega}} + \bar{\Delta}_{\dot{\omega}} + \Delta_\omega + \bar{\Delta}_\omega = 2(\Delta_\phi + 2 + \Delta_\psi + 1) = 0. \tag{13}$$

The physical conditions coming from the inviscid Navier-Stokes equations and the enstrophy flux then prompt us to look for a minimal model $\mathcal{M}_{(p,q)}$ containing two primary fields satisfying the following two conditions:

$$[\psi] \times [\psi] = [\phi] + \text{less singular terms}; \quad \text{and} \quad \Delta_\phi + \Delta_\psi = -3. \tag{14}$$

Polyakov has found the following solution: $\mathcal{M}_{(2,21)}$, with $\psi = \Phi_{(1,4)}$ and $\phi = \Phi_{(1,7)}$. Unfortunately, there are many (perhaps infinitely many) other solutions to these conditions. A very detailed list of solutions with energy spectrum steeper than k^{-3} is given very recently in [5]. We also independently found the first few of them with the help of a simple computer program. In the table below we summarize the first few solutions we have found, with no particular constraints on the energy spectrum. The first one is Polyakov's solution. Together with his solution, $\mathcal{M}_{(3,25)}$, $\mathcal{M}_{(3,26)}$ and $\mathcal{M}_{(6,55)}$ give the most reasonable energy spectrum, i.e., steeper than k^{-3}. These are among those found by Matsuo in [5], where the condition on the steepness of the spectrum is enforced by the extra constraint $\Delta_\psi < -1$. Notice that only the first and the last solutions are parity invariant, in the sense $[\psi] \notin [\psi] \times [\psi]$, but there are others, not included in the table.

First few solutions of Polyakov's requirements

(p,q)	ψ	ϕ	$E(k) \approx$
$(2,21)$	$\Phi_{(1,4)}$	$\Phi_{(1,7)}$	$k^{-25/7}$
$(3,25)$	$\Phi_{(1,11)}$	$\Phi_{(1,9)}$	$k^{-23/5}$
$(3,26)$	$\Phi_{(1,5)}$	$\Phi_{(1,9)}$	$k^{-55/13}$
$(3,35)$	$\Phi_{(1,21)}$	$\Phi_{(1,11)}$	$k^{-9/7}$
$(5,51)$	$\Phi_{(1,17)}$	$\Phi_{(1,11)}$	$k^{-47/17}$
$(6,55)$	$\Phi_{(1,14)}$	$\Phi_{(1,9)}$	$k^{-41/11}$

These results show that the requirements above are not restrictive enough to pin down uniquely one minimal model. Nevertheless, Polyakov's solution is somewhat special in that it contains the smallest number of primary fields and hence might be more stable. It is interesting to include magnetic fields into the picture to see if constraints are more stringent. But before we discuss this attempt, let us first consider a simpler situation: the diffusion of a passive scalar into Polyakov's $\mathcal{M}_{(2,21)}$ solution.

The term "passive scalar" denotes any scalar field such as temperature, density of a pollutant etc..., that does not modify the Navier-Stokes equations and satisfies the diffusion equation

$$\frac{d}{dt}T = \kappa \partial_\alpha \partial_\alpha T. \tag{15}$$

The assumption that T does not appear into the Navier-Stokes equations always implies some approximation. For instance, if T stands for the absolute temperature, the assumption is that the dilation coefficient of the fluid is small, i.e., the density remains constant over the temperature range being examined. From now on we will identify T with the absolute temperature.

We are interested in the so called inertial-convective range, where the Navier-Stokes equation is (7) and the equation for the temperature reduces to $\frac{d}{dt}T = 0$. The temperature spectrum is defined, in analogy with the energy spectrum, as

$$\frac{1}{2} < T^2 >= \int d^1 k E_T(k). \tag{16}$$

Let us first look at the spectrum in the Kolmogorov approximation. Let us define the temperature dissipation rate as ϵ_T. Clearly,

$$\dim(E_T) = \dim(T^2) \times \text{length} \quad \text{and} \quad \dim(\epsilon_T) = \dim(T^2) \times \text{time}^{-1} \tag{17}$$

The temperature spectrum must then be proportional to ϵ_T and, since in the enstrophy dominated inertial range the only other dimensional parameter is η, one obtains the well known scaling law

$$E_T(k) \approx \epsilon_T \eta^{-1/3} k^{-1} \tag{18}$$

This is the Kolmogorov-type prediction of the spectrum in the inertial-convective range. Note, in particular, that it is obeyed by the vorticity ω itself when treated as a passive scalar, being $E_\omega(k) = k^2 E(k)$ and $\epsilon_\omega = \eta$.

The universality of this behavior suggests that, in this regime, the exponent for the temperature spectrum should be the same as that for the vorticity spectrum, even if the latter is not simply given by the Kolmogorov exponent -1 above. The $\mathcal{M}_{(2,21)}$ solution predicts $E_T(k) \approx k^{-11/7}$. In any case, it is not correct to try to identify another primary field in $\mathcal{M}_{(2,21)}$ as the temperature because the predicted exponents would be very far off.

Let us now turn to the main focus of this paper: two dimensional turbulence in presence of magnetic fields. In three dimensional space, the magnetohydrodynamics (MHD) equations can be written, in the limit of infinitely conducting, incompressible and inviscid fluid, purely as functions of the velocity $\vec{v}$ and the magnetic field $\vec{B}$. It is convenient to rescale the magnetic field as given in S.I. units by the constant factor $1/\sqrt{\mu\rho}$, so that the rescaled field has the dimensions of velocity. With this convention, taking the curl of the Navier-Stokes equation to get rid of the pressure term, the equations take the form

$$\begin{aligned} \vec{\nabla} \cdot \vec{v} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \Big(\partial_t \vec{v} \Big) &= \vec{\nabla} \times \Big(- (\vec{v} \cdot \vec{\nabla}) \vec{v} + (\vec{B} \cdot \vec{\nabla}) \vec{B} \Big) \\ \partial_t \vec{B} &= (\vec{B} \cdot \vec{\nabla}) \vec{v} - (\vec{v} \cdot \vec{\nabla}) \vec{B} \end{aligned} \tag{19}$$

Let us now consider the case of an "effectively two dimensional" theory, defined by the condition:

$$\partial_3 \vec{v} = 0 \quad \text{and} \quad \partial_3 \vec{B} = 0. \tag{20}$$

With this assumption, one can treat the third components of the magnetic field and of the velocity as a two dimensional scalar: $B_3 \equiv B$ and $v_3 \equiv v$. Furthermore, one can introduce two stream functions A and ψ for the first two components of $\vec{B}$ and $\vec{v}$ respectively:

$$B_\alpha = \epsilon_{\alpha\beta} \partial_\beta A \quad \text{and} \quad v_\alpha = \epsilon_{\alpha\beta} \partial_\beta \psi. \tag{21}$$

For this effectively two dimensional theory, equations (19) can be written simply in terms of four independent two dimensional scalars: A, B, v and ψ. These equations look particularly nice if one also introduces (as dependent quantities) the two dimensional curls

$$J \equiv \epsilon_{\alpha\beta} \partial_\alpha B_\beta = -\partial_\alpha \partial_\alpha A \quad \text{and} \quad \omega \equiv \epsilon_{\alpha\beta} \partial_\alpha v_\beta = -\partial_\alpha \partial_\alpha \psi, \tag{22}$$

and the differential operator:

$$\mathcal{A} = \epsilon_{\alpha\beta} \partial_\beta A \partial_\alpha. \tag{23}$$

With the aid of the quantities defined above, the equations of motion can than be written as

$$
\begin{aligned}
\frac{d}{dt}\omega &= \mathcal{A}J \\
\frac{d}{dt}v &= \mathcal{A}B \\
\frac{d}{dt}B &= \mathcal{A}v \\
\frac{d}{dt}A &= 0.
\end{aligned}
\tag{24}
$$

This shows that the fields B and v only appear in the second and the third equations of (24). It is therefore consistent to study first the possible turbulent solution of A and ψ by setting B and v to zero.

Before we go any further, one important remark must be made, in order to avoid confusion. The kind of turbulent solution we are considering is not the same as the kind of two dimensional turbulence that has recently been studied in laboratory [6]. In laboratory experiments, one is dealing with magnetic Reynolds numbers that are fairly small and the system (typically a tank filled with mercury) is subjected to a strong external magnetic field perpendicular to the plane. One then writes the total magnetic field as the sum of the external one $\vec{B}_{\mathrm{ext.}}$ and the turbulent one $\vec{b}$. Since the turbulent component is much smaller than the external field, the MHD equations can be expanded in first order in $\vec{b}$. After taking the curl, the magnetic field drops out completely of the Navier-Stokes equation and its only effect is to reduce the $\vec{B} = 0$ three dimensional Navier-Stokes equation to the $\vec{B} = 0$ two dimensional Navier-Stokes equation (5). One could then study the behavior of $\vec{b}$ as a passive vector on the ordinary turbulent background. As far as conformal field theory goes, one is then brought back to Polyakov's solution.

In the following, we consider a different regime altogether. We assume that there is no external magnetic field and that the turbulent magnetic field is parallel to the plane and is (in our units) comparable with the velocity. This can happen only at very high magnetic Reynolds number, such as in astrophysical situations (for instance in a galaxy). One last word of caution: we are not interested in the dynamo effect. As a matter of fact, the dynamo effect does not take place at all in two dimensions [7]. In two dimensions, in order to keep turbulence in the stationary regime, one has to pump in magnetic energy as well as kinetic energy, otherwise the magnetic field would dissipate due to the finite conductivity of the material just as much as the velocity would dissipate due to the finite viscosity.

The third component of the magnetic field need not be ignored but can be treated at a later stage since it decouples from the equations for ψ and A as already stressed. In any case, one cannot invoke this magnetic field as the agent that blocks the system into the two dimensional state. However, if one is willing to take the astrophysical scenario seriously, there are other possible effects that might help, such as the rotation of the galaxy or gravity itself.

Let us start by rewriting the equations for ψ and A more explicitly

$$\begin{aligned} &\dot{\omega} + \epsilon_{\alpha\beta}\partial_\beta\psi\partial_\alpha\omega = \epsilon_{\alpha\beta}\partial_\beta A\partial_\alpha J \\ &\dot{A} + \epsilon_{\alpha\beta}\partial_\beta\psi\partial_\alpha A = 0. \end{aligned} \tag{25}$$

From (25) we see that a radical change takes place in this regime. Enstrophy is no longer a conserved quantity. Its place is taken by the integral of the square of the stream function A. The only two quadratic conserved quantities are then

$$E = \frac{1}{2}\int (v_\alpha v_\alpha + B_\alpha B_\alpha) d^2x \quad \text{and} \quad G = \frac{1}{2}\int A^2 d^2x. \tag{26}$$

It is then natural to look for a solution for which

$$< \dot{A}(r)A(0) > \approx r^0. \tag{27}$$

Instead of the constant enstrophy condition, we are considering ϵ_A, the dissipation rate of A, as a constant.

Let us translate these physical conditions in the language of conformal field theory. Again, the working assumption is that we try to fit the requirements (25) and (27) by a minimal model. We interpret the two basic scalars A and ψ as primary fields and $\dot{A}$ as a secondary field in the conformal family $[\chi]$ of the most singular field appearing in the fusion rules of A and ψ. By using point split regularization, one can see that $\dot{A}$ must be a level two scalar in $[\chi]$.

We must then look for a minimal model $\mathcal{M}_{(p,q)}$ satisfying the following requirements:

a) There are two primary fields A and χ for which $\Delta_A + \Delta_\chi = -2$.

b) There is another primary field ψ such that its fusion rules are, up to less singular terms, $[\psi] \times [A] = [\chi]$.

To be sure, these conditions do not fix the theory uniquely. The simplest solution however, provides a nice example. Surprisingly, this solution is simpler (i.e., contains less primary fields) than Polyakov's. It is the minimal model $\mathcal{M}_{(2,13)}$, whose basic structure is summarized in the table below.

Structure and interpretation of $\mathcal{M}_{(2,13)}$

	$\Phi_{(1,1)}$	$\Phi_{(1,2)}$	$\Phi_{(1,3)}$	$\Phi_{(1,4)}$	$\Phi_{(1,5)}$	$\Phi_{(1,6)}$
meaning:	I	ψ	?	A	χ	?
$\Delta =$	0	$-5/13$	$-9/13$	$-12/13$	$-14/13$	$-15/13$

It should not be surprising that this solution is not the same as the one before. Remember that the introduction of magnetic fields adds an extra dimensional parameter to the theory. This parameter is still relevant in the inertial range and it affects the scaling laws already in Kolmogorov theory.

Having found a solution for our physical requirements, it is now a simple exercise in conformal field theory to work out its basic predictions. The kinetic energy and magnetic energy spectra behave respectively as

$$E_{\text{kin.}}(k) \approx k^{-7/13} \quad \text{and} \quad E_{\text{mag.}}(k) \approx k^{-35/13}. \tag{28}$$

This means that the equipartition theorem does not hold exactly for our solution. In fact, the kinetic energy will dominate at short distance and the magnetic one will take over at large distance.

Passive scalars will still be represented by level one fields. But now, the argument given above using the vorticity no longer apply, because ω no longer satisfies the equation

$\frac{d}{dt}\omega = 0$. For example, if one takes $T \in [\Phi_{(1,6)}]$, level one scalar, one finds $E_T(k) \approx k^{-21/13}$ but we do not see any physical reason for that identification.

In conclusion, Polyakov's idea gives a fresh new look on one of the oldest unsolved problems of physics. Unfortunately, the physical requirements found so far are not enough to uniquely identify one specific minimal model. In fact, there are perhaps infinitely many minimal models satisfying the constraints. One wonders whether the "correct" conformal field theory might be the one obtained as some limiting case of the minimal models above or perhaps as some other, non minimal, conformal field theory. Quite apart from this, the introduction of magnetic fields and passive scalars is of some interest. It is still not enough to fix the theory completely but the simplest model satisfying the physical requirements gives sensible predictions that might be relevant in the astrophysical context.

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